Sorting Effects of Performance Pay
Web Appendix

September 4, 2013

Abstract

This is the online appendix for the paper Sorting Effects of Performance Pay by Korok Ray and Maris Goldmanis, published in Management Science 2013. This appendix includes all omitted materials from the paper, such as extra proofs and figures. Any time in the original paper that we say that the details are available on request, this web appendix has the details.

1 Numerical Simulations

We conducted several numerical simulations behind the corollaries. Figure 1 gives the numerical simulations behind Corollary 1 in the published manuscript. Figure 2 gives the numerical simulations behind Corollary 2 in the published manuscript. Figure 3 gives the numerical simulations behind Corollary 3 in the published manuscript.

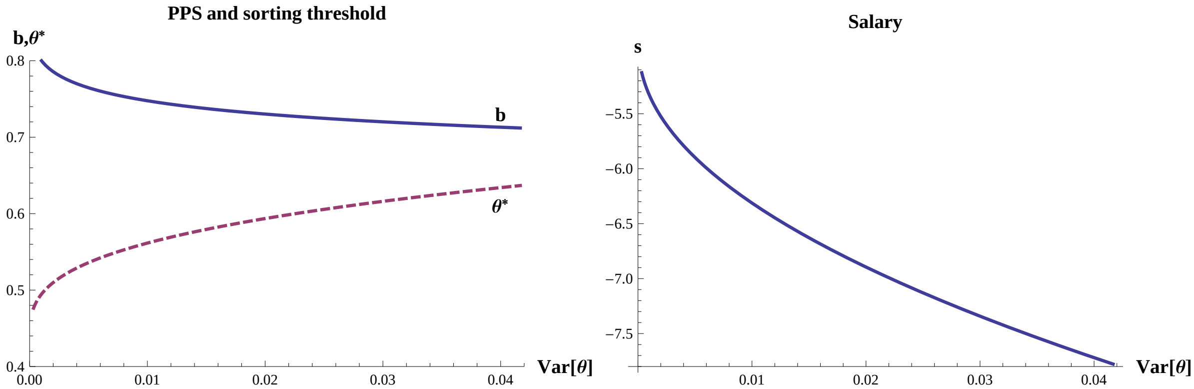


Figure 1: Comparative statics with respect to the variance of manager type. Numerical simulations using uniform distributions with parameters $\mu = 0.5$, $\bar{u} = 2.5$, $\gamma = 1$, $k = 5$, $m = 1$, and $c = 0.01$.

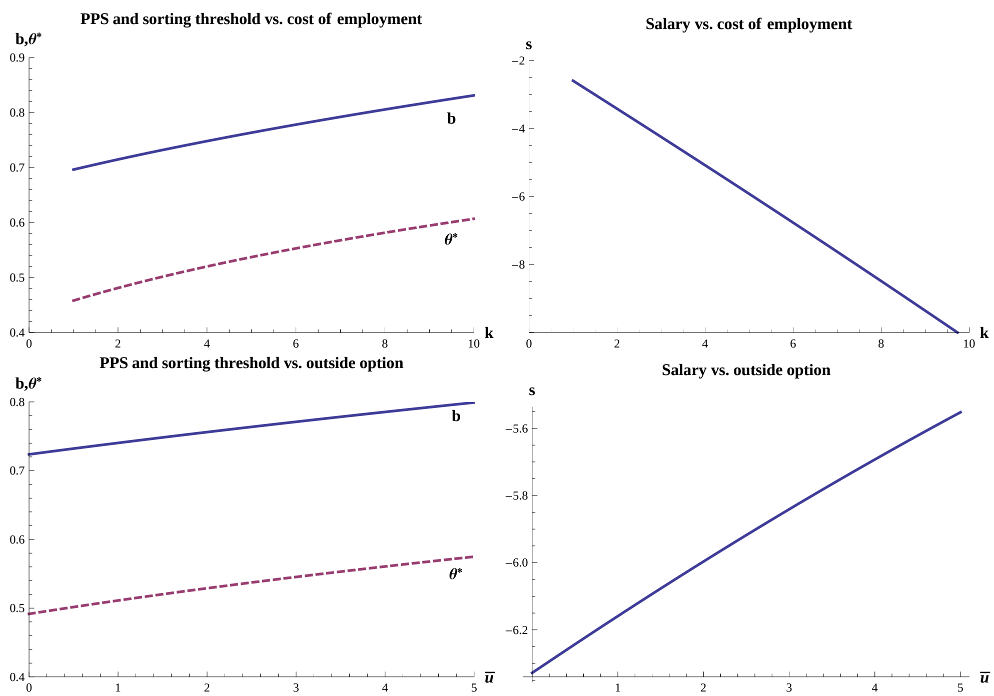


Figure 2: Comparative statics with respect to the manager's cost of employment and outside option. Numerical simulations using a uniform distribution with parameters $\mu = 0.5$, $a = 0.25$, $\bar{u} = 2.5$ (for varying k), $\gamma = 1$, $k = 5$ (for varying $\bar{u}$), $m = 1$, and $c = 0.01$.

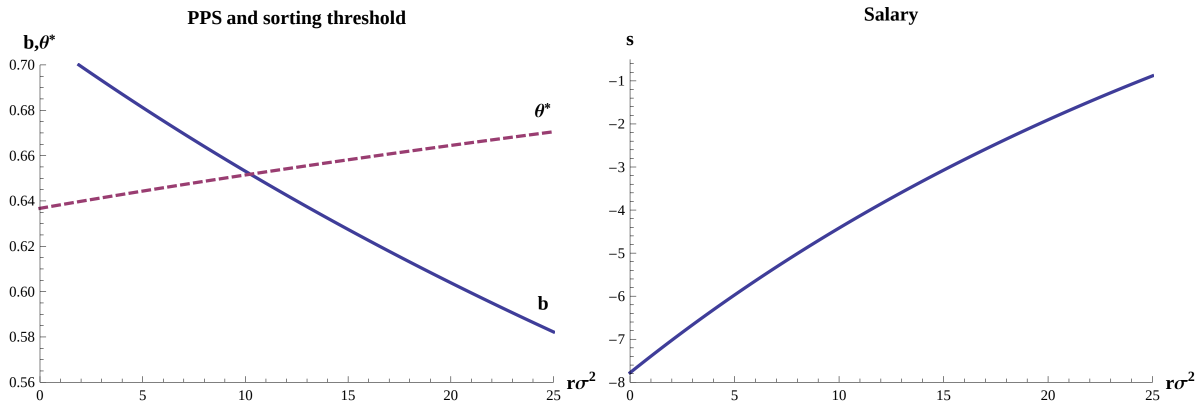


Figure 3: Comparative statics with respect to the manager's level of risk aversion and output variability. Numerical simulations using a uniform distribution on $[0, 1]$ with parameters $\bar{u} = 2.5$, $\gamma = 1$, $k = 5$, $m = 1$, and $c = 0.01$.

2 Increasing outside options

2.1 Functional form

2.1.1 No productive effort

Suppose we are in the baseline model, where the manager does not exert productive effort. Recall that in this model output is proportional to ability: $x \sim \theta$, while total costs are equal to the firm's employment costs, which are decreasing linearly with ability ($C = C_{\mathrm{f}} = k - m\theta$). Since both output and total costs are affine functions of ability, so is the total surplus.

Assuming that similar production technologies are available at all firms, the surplus generated by the manager at any rival firm will also be an affine function of (i.e., increase linearly with) ability. Given similar division of the surplus at all firms, it is natural to expect that the share of the surplus left for the manager at these firms (which determines the manager's outside option at the current firm) will also increase linearly with ability. We assume the simplest such functional dependence:

$$\bar{u}(\theta) = \bar{u}_0 + \beta\theta.$$

2.1.2 Productive effort

Now, suppose we are in the full model with productive effort. Recall that by the assumptions we made about the production function at the current firm, output is directly proportional to ability, θ, and effort e, while effort (in equilibrium) is directly proportional to ability: $x \sim \theta \cdot e$; $e \sim \theta$. It follows that equilibrium output is proportional to the *square* of ability: $x \sim \theta^2$. In addition, the firm's employment cost is decreasing linearly with ability ($C_{\text{f}} = k - m\theta$), while the manager's cost of effort is proportional to the square of effort, and hence to the square of ability ($C_{\text{m}} \sim e^2 \sim \theta^2$). Total costs are therefore a quadratic function of ability. Since both output and total costs are quadratic functions of ability, so is the total surplus.

Assuming that similar production technologies are available at all firms, the surplus generated by the manager at any rival firm will also be a quadratic function of ability. Given similar division of the surplus at all firms, it is natural to expect that the share of the surplus left for the manager at these firms (which determines the manager's outside option at the current firm) will also be a quadratic function of ability. We assume the simplest such functional dependence:

$$\bar{u}(\theta) = \bar{u}_0 + \beta\theta^2.$$

2.2 Results

2.2.1 No productive effort

The expected total surplus is now $E[TS|\theta] = (\gamma + m - \beta)\theta - (k + \bar{u}_0)$. Since we are interested in the cases where positive sorting is optimal, we must assume $\beta < \gamma + m$. Under this assumption, the first-best positive sorting threshold is

$$\theta^{FB} = \frac{k + \bar{u}_0}{\gamma + m - \beta} > 0.$$

The firm's conditional profit function is, as before (i.e., as with ability-independent outside options), $E[\pi|\theta] = \gamma\theta(1 - b) - s - (k - m\theta)$. The manager's surplus is now

$$E[MS|\theta] = s + b\gamma\theta - \bar{u}_0 - \beta\theta = \left(b - \frac{\beta}{\gamma}\right)\gamma\theta + s - \bar{u}_0.$$

The manager's participation decision thus takes the same form as before, only with $\hat{b} = b - \beta/\gamma$ in place of b, so that the firm's objective is to

$$\max\Big\{\max_{b\gamma-\beta>0;s}\int_{\max(\theta^*(s,b),0)}^{1} E[\pi|\theta]f(\theta)d\theta;$$
$$\max_{b\gamma-\beta<0;s}\int_{0}^{\min(\theta^*(s,b),1)} E[\pi|\theta]f(\theta)d\theta;$$
$$\int_0^1 E[\pi(b = \beta/\gamma; s = \bar{u}_0)|\theta]f(\theta)d\theta;\ 0\Big\},$$

which is identical to the objective with ability-independent outside options (stated at the beginning of the proof of Proposition 1), only with $b\gamma - \beta$ in place of b in the definition of the four cases to consider and in the formula for $\theta^*(s, b)$.

The proof of Proposition 1 now carries through virtually unchanged, delivering the following modified version of Proposition 1:

Proposition 1': In the pure sorting model, the optimal contract consists of $b = \beta/\gamma + \epsilon_b$ and $s = \bar{u}_0 - \epsilon_s$ and induces $\theta^* = \theta^{FB} + \epsilon_\theta$, where ϵ_b, ϵ_s, and ϵ_θ are infinitesimal positive quantities.

2.2.2 Productive effort

Total surplus with first-best effort is now

$$E[TS^*|\theta] = \frac{(\gamma\theta)^2}{2c} - (k - m\theta) - \bar{u}_0 - \beta\theta^2.$$

Since we are interested in the cases where positive sorting is optimal, we must assume that this is everywhere increasing. A sufficient condition for that is that $\gamma^2/c - 2\beta + m > 0$. With this assumption, the first-best positive sorting threshold is

$$\theta^{FB} = \frac{cm - \sqrt{c\left(c\left(m^2 - 4(k + \bar{u}_0)\beta\right) + 2(k + \bar{u}_0)\gamma^2\right)}}{2c\beta - \gamma^2}.$$

Finally, we assume (as before) that $0 < \theta^{FB} < 1$.

The manager's expected surplus is now

$$E[MS|\theta] = s + b\gamma\theta e - \bar{u}_0 - \beta\theta^2 - \frac{1}{2}ce^2.$$

Optimal effort is, as before, $e^* = b\gamma\theta/c$, so that the manager's surplus at optimal effort is

$$E[MS|\theta] = \left[\frac{(b\gamma)^2}{2c} - \beta\right]\theta^2 - \bar{u}_0 + s.$$

The manager's participation decision takes the same form as before (albeit with different thresholds): if $(b\gamma)^2/(2c) = \beta$, participate if and only if $s \geq \bar{u}_0$, if $(b\gamma)^2/(2c) < \beta$, participate if and only if $\theta \leq \theta^*$; if $(b\gamma)^2/(2c) > \beta$, participate if and only if $\theta \geq \theta^*$, where

$$\theta^* = \sqrt{\frac{2c(\bar{u}_0 - s)}{(b\gamma)^2 - 2c\beta}}.$$

As before, the firm's optimal profit in equilibrium is

$$E[\pi|\theta] = (\gamma\theta)^2 b(1 - b)/c - s - (k - m\theta).$$

Because the profit is strictly decreasing in salary, the firm will always choose the minimum salary that satisfies individual rationality for all manager types hired, i.e., one that gives the marginal manager type zero surplus:

$$s = \bar{u}_0 - \left[\frac{(b\gamma)^2}{2c} - \beta\right](\theta^*)^2.$$

The firm's profit for a given manager type thus becomes

$$E[\pi|\theta] = \frac{\gamma^2\theta^2 b(1 - b)}{c} - \bar{u}_0 + \left[\frac{b^2\gamma^2}{2c} - \beta\right](\theta^*)^2 - k + m\theta,$$

Note that the derivative of this conditional profit function with respect to b is the same as with constant outside options,

$$\frac{\partial E[\pi|\theta]}{\partial b} = \frac{\gamma^2\left((1-2b)\theta^2 + b(\theta^*)^2\right)}{c},$$

while the derivative with respect to the sorting threshold has changed to

$$\frac{\partial E[\pi|\theta]}{\partial \theta^*} = 2\left(\frac{b^2\gamma^2}{2c} - \beta\right)\theta^*.$$

The firm's problem can now be described as follows:

$$\max\Big\{\max_{\theta^*;b:(b\gamma)^2/(2c)>\beta}\int_{\theta^*}^1 E[\pi|\theta]f(\theta)d\theta;$$
$$\max_{\theta^*;b:(b\gamma)^2/(2c)<\beta}\int_0^{\theta^*} E[\pi|\theta]f(\theta)d\theta;$$
$$\int_0^1 E[\pi(b=\sqrt{2c\beta}/\gamma;\theta^*=0)|\theta]f(\theta)d\theta;\ 0\Big\}.$$

Since total surplus is strictly increasing in θ by assumption, arguments similar to those in the proof of Proposition 1 show that negative sorting or no sorting cannot be optimal. Hence the firm's problem reduces to

$$\max_{\theta^*;b:(b\gamma)^2/(2c)>\beta}\int_{\theta^*}^1 E[\pi|\theta]f(\theta)d\theta.$$

Unlike in the main model with constant outside options, we now have a possibly binding constraint that ensures positive sorting: $(b\gamma)^2/(2c) > \beta$. (In fact, it is easy to verify that the constraint binds for sufficiently high values of β.) Letting ϵ be the lowest positive value that $(b\gamma)^2/(2c)-\beta$ can attain given the finite divisibility of money, we can replace the strict inequality constraint with a weak inequality constraint and apply the Kuhn-Tucker conditions to the resuting standard constrained optimization problem. The Lagrangian for this problem is

$$\mathcal{L} = \int_{\theta^*}^1 E[\pi|\theta]f(\theta)d\theta + \lambda((b\gamma)^2/(2c) - \beta - \epsilon).$$

The first-order conditions for maximum are

$$0 = -E[\pi|\theta^*] + \int_{\theta^*}^1 \frac{\partial E[\pi|\theta]}{\partial \theta^*}f(\theta)\,d\theta$$

and

$$0 = \int_{\theta^*}^1 \frac{\partial E[\pi|\theta]}{\partial b}f(\theta)\,d\theta + \lambda b\gamma/c;\quad \lambda((b\gamma)^2/(2c)-\beta-\epsilon)=0;\quad \lambda \geq 0;\quad (b\gamma)^2/(2c)-\beta \geq \epsilon.$$

These conditions provide an exact analogue for **Proposition 2**.

Furthermore, because at the optimum $(b\gamma)^2/(2c) > \beta$, we have

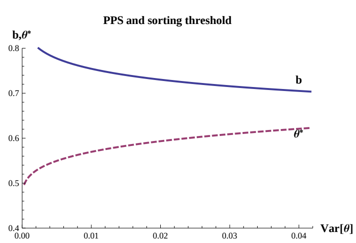
$$\frac{\partial E[\pi|\theta]}{\partial \theta^*} = 2\left(\frac{b^2\gamma^2}{2c} - \beta\right)\theta^* > 0 \quad\Rightarrow\quad \int_{\theta^*}^1 \frac{\partial E[\pi|\theta]}{\partial \theta^*}f(\theta)\,d\theta > 0,$$

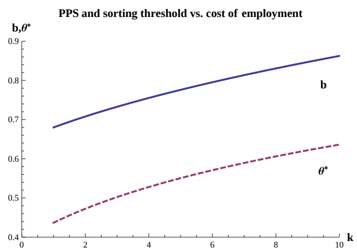
so that by the first-order condition with respect to θ^*,

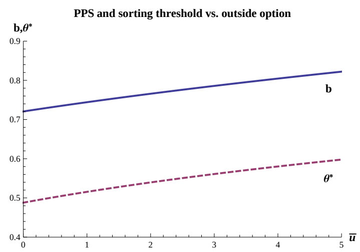
$$E[\pi|\theta^*] = \int_{\theta^*}^{1} \frac{\partial E[\pi|\theta]}{\partial \theta^*} f(\theta)\, d\theta > 0,$$

Now, the logic of the original **Proposition 3** applies, proving that $\theta^* > \theta^{\mathrm{FB}}$. To repeat the argument for completeness: Denoting the manager's surplus by $E[MS|\theta]$, we note that, by definition, $E[\pi|\theta] + E[MS|\theta] = E[TS|\theta]$: the sum of the firm's and the manager's surpluses equals total surplus for any θ. Evaluating this at $\theta = \theta^*$ and noting that, by definition, $E[MS|\theta^*] = 0$, we obtain $E[TS|\theta^*] = E[\pi|\theta^*] > 0$. Furthermore, noting that the total surplus actually achieved cannot exceed its first-best level (obtained by setting $b = 1$), we must conclude that $E[TS^*|\theta^*] > 0 = E[TS^*|\theta^{FB}]$, where TS^* is the total surplus with first-best effort level. Since $E[TS^*|\theta]$ is strictly increasing in θ for all $\theta > 0$, this proves that $\theta^* > \theta^{FB}$.

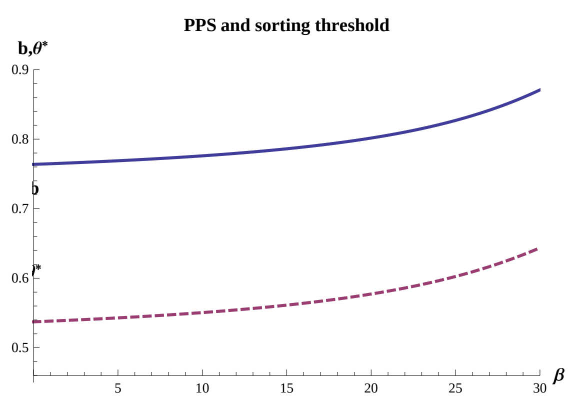
The comparative statics results in Corollaries 1 and 2 continue to hold in numerical examples with increasing outside options (we have no analytical proofs). Using the same parameter values as in the numerical examples in the paper (in addition to $\beta = 10$), we obtained graphs that are almost identical to those in the paper:







We also observe that both the bonus and the sorting threshold increase as outside options increase more steeply (β increases):



3 Alternative Specification of Output

Following the referee's suggestion, suppose that output is given by

$$x = (1 + \gamma\theta)e + \epsilon \tag{1}$$

Expected output is:

$$E[x \mid \theta] = (1 + \gamma\theta)e \tag{2}$$

Wages are

$$w = s + bx \tag{3}$$

Expected wage is

$$E[w \mid \theta] = s + b(1 + \gamma\theta)e \tag{4}$$

Expected total surplus is

$$E[TS \mid \theta] = (1 + \gamma\theta)e - C(e) - (k - m\theta) \tag{5}$$

The first best level of effort maximizes total surplus

$$e^{FB} = (1 + \gamma\theta)/c \tag{6}$$

The cost of effort function is

$$C(e) = \frac{c}{2}e^2 \tag{7}$$

At $e^{FB}, E[TS] = \frac{(1+\gamma\theta)^2}{2c} - (k - m\theta)$. Expected total surplus is positive if

$$E[TS \mid \theta] > 0 \Longleftrightarrow (1 + \gamma\theta)^2 > (k - m\theta)2c \tag{8}$$

Rearranging and simplifying, this occurs if

$$\gamma^2\theta^2 + 2(\gamma + cm)\theta + (1 - 2ck) > 0 \tag{9}$$

θ^{FB} is the solution when the second order polynomial above equals 0, or

$$\theta^{FB} = \frac{-2(\gamma + cm) \pm \sqrt{2(\gamma - cm)^2 - 4\gamma^2(1 - 2ck)}}{2\gamma^2} \tag{10}$$

The agent maximizes his expected payoff less his cost of effort

$$\max_{e} E[w \mid \theta] - C(e) \tag{11}$$

This yields the agent's incentive constraint

$$\hat{e} = b(1+\gamma\theta)/c \tag{12}$$

The agent's binding participation constraint is

$$(PC)E[\hat{w} \mid \theta^*] - C(\hat{e}) = \bar{u} \tag{13}$$

Plugging in the incentive constraint and rewriting gives

$$s + \frac{b^2(1+\gamma\theta)^2}{2c} = \bar{u} \tag{14}$$

The marginal manager θ^* solves this equation. Rearranging, this gives

$$\theta^* = \frac{\sqrt{\frac{(\bar{u}-s)2c}{b^2}} - 1}{\gamma} \tag{15}$$

$$= \sqrt{\frac{2c(\bar{u}-s)}{b\gamma}} - \frac{1}{\gamma} \tag{16}$$

Let $\tilde{\theta} = \frac{\sqrt{2c(\bar{u}-s)}}{b\gamma}$. So $\theta^* = \tilde{\theta} - \gamma^{-1}$. Observe that $\tilde{\theta}$ is the marginal manager from the original model of $x = \gamma\theta e + \epsilon$. θ^* is a linear transformation of $\tilde{\theta}$. Observe further that the derivatives with respect to the contract parameters are the same.

$$\frac{\partial\theta^*}{\partial s} = \frac{\partial\tilde{\theta}}{\partial s} = \frac{-c}{(b\gamma)^2\tilde{\theta}} \text{ and } \frac{\partial\theta^*}{\partial b} = \frac{\partial\tilde{\theta}}{\partial b}\frac{-\tilde{\theta}}{b} \tag{17}$$

$$\frac{\partial\theta^*}{\partial b} = \frac{\partial\tilde{\theta}}{\partial b} = \frac{\partial\tilde{\theta}}{\partial s}\frac{b\tilde{\theta}^2\gamma^2}{c} = \frac{\partial\theta^*}{\partial s}\frac{b(\theta^*+\gamma^{-1})^2\gamma^2}{c} \tag{18}$$

And the firm's expected profit is given by

$$E[\pi \mid \theta] = (1+\gamma\theta)\hat{e} - [s + b(1+\gamma\theta)\hat{e}] - (k - m\theta) \tag{19}$$

$$= (1-b)b(1+\gamma\theta)^2/c - s - (k - m\theta) \tag{20}$$

The derivatives of expected profit with respect to the contract parameters are

$$\frac{\partial E[\pi \mid \theta]}{\partial s} = -1 \tag{21}$$

$$\frac{\partial E[\pi \mid \theta]}{\partial b} = \frac{(1+\partial\theta)^2}{c}(1-2b) \tag{22}$$

Plugging these into the first order condition from the firm's problem gives

$$-\frac{\partial\theta^*}{\partial b}(1-F(\theta^*)) = \frac{\partial\theta^*}{\partial s}\int_{\theta^*}^{1}\frac{(\gamma\theta)^2}{c}(1-2b)f(\theta)d\theta \tag{23}$$

Substituting in $\tilde{\theta} = \theta^* + \gamma^{-1}$

$$\left(-\frac{b\tilde{\theta}^2\gamma^2}{c}\right)(1-F(\theta^*)) = \int_{\theta^*}^{1}\frac{(\gamma\theta)^2}{c}(1-2b)f(\theta)d\theta \tag{24}$$

Rearranging and solving for b gives

$$b = \left(2 - \frac{\theta^* + \gamma^{-1}}{E[\theta^2 \mid \theta > \theta^*]}\right)^{-1} \tag{25}$$

Once again, substituting $\tilde{\theta} = \theta^* + \gamma^{-1}$

$$b = \left(2 - \frac{\tilde{\theta}^2}{E[\theta^2 \mid \theta > \theta^*]}\right)^{-1} \tag{26}$$

The first order condition with respect to s gives

$$\left[\frac{(1+\gamma\theta^*)^2}{c}b(1-b) - s - (k - m\theta)\right] f(\theta^*)\frac{c}{(b\gamma)^2\tilde{\theta}} = 1 - F(\theta^*) \tag{27}$$

Rearranging and solving for s, substituting θ^* for $\tilde{\theta}$:

$$s = \frac{(1+\gamma\theta^*)^2}{c}b(1-b) - (k - m\theta) - \frac{1-F(\theta^*)}{f(\theta^*)}\frac{b^2\gamma^2}{c}(\theta^* + \gamma^{-1}) \tag{28}$$

It is clear from these expressions that the optimal contract s and b can be written either in terms of $\tilde{\theta}$ or θ^*. The contract parameters are largely similar.

4 Non-Linear Contracts

Suppose the firm offers a (potentially non-linear) contract $w(x)$. The linear contract in the manuscript is a special case where $w(x) = s + bx$.

Observe that the linear contract in Proposition 1 achieves first-best and extracts all the surplus, therefore a non-linear contract cannot do better. Indeed, there is no second-best problem in the benchmark model without incentives, so in Section 2 a non-linear contract provides no extra gain for the firm over a linear contract.

4.1 General Non-Linear Contracts in Section 3

In Section 3 of the paper, under incentives, a non-linear contract may do better. A manager θ earns expected wage

$$E[w(x)|\theta] = \int_{-\infty}^{\infty} w(\gamma\theta e + \epsilon)g(\epsilon)d\epsilon$$

where ϵ follows the symmetric distribution g that has a mean 0 and variance σ^2. Let G be its CDF. Because the contract is non-linear, it is no longer possible to swap the integral and $w(\cdot)$. To get traction, a unique cutoff θ^* exists only if the expected profit function is increasing in θ. Assume this is true, i.e. consider a contract $w(\cdot)$ such that $E[w(x)|\theta]$ increases in θ. The marginal manager θ^* will be given by

$$E[w(x)|\theta^*] = \bar{u}$$

This is defined implicitly, since the wage function can take any form. Expected profits for each θ are

$$E[\pi|\theta] = \gamma\theta e - E[w(x)|\theta] - (k - m\theta)^+$$

The firm maximizes expected profit:

$$\max_{w(\cdot)} \int_{\theta^*}^{\infty} E[\pi|\theta]f(\theta)d\theta.$$

The contract space is much larger, since the firm maximizes over functions, rather than two parameters. This, in general, does not yield closed-form solutions, since the wage function w is embedded both in the expected profit function, as well as in θ^*. The non-linearity of the wage function prevents explicit (or even implicit) solutions, and therefore this optimization is intractable.

5 Bonus and Target Contracts

Suppose the firm offers a bonus B if output clears some pre-specified target $\bar{x}$. In this case, the contract is a pair $(B, \bar{x})$. Therefore, this class of contracts yields

$$w(x) = \begin{cases} B & \text{if } x \geq \bar{x} \\ 0 & \text{if } x < \bar{x} \end{cases}$$

This is a subclass of nonlinear contracts. It is an alternative incomplete contract to the linear contract.

5.1 A Bonus and a Target in Section 2

Consider Section 2, without moral hazard. The linear contract achieves first best, it remains to show if the bonus and target can as well. The expected wage to the manager of type θ is

$$\begin{aligned} E[w(x)|\theta] &= BPr(x > \bar{x}) \\ &= B(1 - G(\bar{x} - \gamma\theta)) \\ &= BG(\gamma\theta - \bar{x}) \end{aligned}$$

where g is the symmetric PDF of ϵ, with mean 0 and variance σ^2, and G is the CDF of ϵ. Expected wage is monotonic in θ, so there exists a θ^* such that $E[w(x)|\theta^*] = \bar{u}$. Solving for θ^*, this is

$$\theta^*(B, \bar{x}) = \frac{G^{-1}(\bar{u}/B) + \bar{x}}{\gamma}$$

To implement first best, $\theta^* = \theta^{FB}$, then

$$\theta^* = \frac{G^{-1}(\bar{u}/B) + \bar{x}}{\gamma} = \theta^{FB}$$

Then $\bar{x}$ satisfies, for each B:

$$\bar{x}(B) = \gamma\theta^{FB} - G^{-1}(\bar{u}/B).$$

Expected profit for each θ is

$$E[\pi|\theta] = \gamma\theta - E[w|\theta] - (k - m\theta)^+$$

To implement first best, pick a contract such that $\theta^*(B, \bar{x}) = \theta^{FB}$. Let ES be the (efficient) set of such contracts. The firm chooses to maximize ex-ante profits, so it solves

$$\max_{(B,\bar{x}) \in ES} \int_{\theta^*(B,\bar{x})}^{\infty} E[\pi|\theta] dF$$

From the expression for $E[\pi|\theta]$, it is clear that solving this optimization is identical to minimizing $E[w|\theta]$. All contracts in the efficient set implement first best. Choosing such a contract reduces to selecting B alone, since $\bar{x}(B)$ is the target that implements θ^{FB} for any B. Thus, the firm minimizes $E[w|\theta]$ for each B, or

$$\min_B BG(\gamma\theta - \bar{x}(B))$$

Assuming the second order conditions hold, the FOC implicitly gives the optimal bonus.

$$B^* = \frac{G(\gamma\theta - \bar{x}(B^*))}{g(\gamma\theta - \bar{x}(B^*))\bar{x}'(B^*)}$$

This optimal bonus B^*delivers an optimal target ($\bar{x}(B^*)$, and implements first best and minimizes expected wages. Note that this is not a corner solution, but rather an interior solution. Although it is difficult to know precisely because of the implicit solution, it is unlikely that this optimal bonus and target minimizes the manager's information rent, as the linear contract does. If so, the firm would prefer the optimal linear contact over the optimal bonus and target, even though both contracts are efficient.

5.2 A Bonus and a Target in Section 3

Now add moral hazard, as in Section 3. Expected wage for each θ in this section is

$$\begin{aligned} E[w(x)|\theta] &= BPr(x > \bar{x}) \\ &= BPr(\epsilon > \bar{x} - \gamma\theta e) \\ &= B(1 - G(\bar{x} - \gamma\theta e)) \\ &= BG(\gamma\theta e - \bar{x}) \end{aligned}$$

where the last equality follows because G is symmetric. It is easy to show that the $E[w(x)|\theta]$ increases in θ. Thus, the marginal manager θ^* is indifferent between his inside and outside options. Thus, θ^* is defined by

$$E[w(x)|\theta^*] - C(\hat{e}) = \bar{u}$$

Rearranging and solving for θ^* yields

$$\theta^* = \frac{G^{-1}\left(\frac{\bar{u}+C(\hat{e})}{B}\right) + \bar{x}}{\gamma\hat{e}}$$

The manager maximizes expected wage less his cost of effort, so he solves

$$\max_e E[w(x)|\theta] - C(e)$$

Taking FOC yields the equilibrium effort, which is now defined implicitly:

$$\hat{e} = \frac{B\gamma\theta}{c} g(\gamma\theta\hat{e} - \bar{x}).$$

The second-order sufficient condition for the manager's problem shows

$$\frac{B\gamma^2\theta^2}{c} g'(\gamma\theta\hat{e} - \bar{x}) < 1 \tag{29}$$

By the implicit function theorem,

$$\frac{\partial \hat{e}}{\partial B} = \frac{\frac{\gamma\theta}{c} g(\gamma\theta\hat{e} - \bar{x})}{1 - \frac{B\gamma^2\theta^2}{c} g'(\gamma\theta\hat{e} - \bar{x})} > 0.$$

The denominator is positive by (29). And,

$$\frac{\partial e}{\partial \bar{x}} = \frac{-\frac{B\gamma\theta}{c} g'(\gamma\theta\hat{e} - \bar{x})}{1 - \frac{B\gamma^2\theta^2}{c} g'(\gamma\theta\hat{e} - \bar{x})}$$

Though the denominator is positive, the numerator can be either positive or negative, depending on the g. Therefore it is difficult to know the sign of the above derivative.

Expected profit of the firm is output less expected wages less the fixed cost. This is

$$E[\pi|\theta] = \gamma\theta\hat{e} - BG(\gamma\theta\hat{e} - \bar{x}) - (k - m\theta)^+$$

The firm maximizes ex-ante profits. It selects a bonus and target to maximizes ex-post profits for every $\theta > \theta^*$. The firm solves

$$\max_{(B,\bar{x})} \int_{\theta^*(B,\bar{x})}^{\infty} E[\pi|\theta] f(\theta) d\theta$$

Combining the FOC from this optimization yields the equilibrium condition:

$$\frac{\partial \theta^*}{\partial B} \int_{\theta^*}^{\infty} \frac{\partial E[\pi|\theta]}{\partial \bar{x}} dF = \frac{\partial \theta^*}{\partial \bar{x}} \int_{\theta^*}^{\infty} \frac{\partial E[\pi|\theta]}{\partial B} dF$$

To solve for the optimal B and $\bar{x}$, it is necessary to expand each of these terms, both inside and outside the integrals. Unfortunately, this is extremely cumbersome. Because the incentive constraint for the manager does not yield an expression for equilibrium effort in closed form, the expressions for all subsequent derivatives are all highly involved and implicit. You can see this from the expression for θ^*. Taking the derivative with respect to B is quite involved, because θ^* is a function of B and a function of $\hat{e}$, which itself is a function of B. Similarly, differentiating the expected profit with respect to B or $\bar{x}$ is also cumbersome, since it is a function of $\bar{x}$, B, and $\hat{e}$. I was not able to simplify the expressions to arrive at the optimal B or $\bar{x}$.

6 Ability-Contingent Outside Options

In the original paper, the outside options $\bar{u}$ do not vary with the manager's type θ. I now explore an extension mentioned in Footnote 7: the result of that section is robust when outside options are increasing and linear in θ.

Suppose that the manager's outside options are now $\bar{u} + \beta\theta$, where $\beta > 0$. Now, $\bar{u}$ is the intercept and β is the slope of the manager's outside options, which are now linear and increasing functions of θ. The marginal manager will set $E[w|\theta^*] = \bar{u} + \beta\theta^*$. Solving for θ^*, this gives

$$\theta^* = \frac{\bar{u} - s}{b\gamma - \beta}.$$

In order for θ^* to be well-defined ($\theta^* > 0$), it must be the case that the numerator and denominator are both positive ($\bar{u} > s$ and $b\gamma > \beta$) or both negative ($\bar{u} < s$ and $b < \beta$). I assume these as regularity conditions. The rest of the development of the model is the same, namely, the setup of the firm's problem. The firm will choose a salary and bonus to maximize expected profits:

$$\max_{s,b} \int_{\theta^*(s,b)}^{\infty} E[\pi|\theta] f(\theta) d\theta$$

The term β is embedded inside θ^*. Solving the firm's problem generates a result akin to Proposition 1 in the paper:

Proposition A1 *In the pure sorting model with ability-contingent outside options, the optimal contract has $s \approx \bar{u}$ and $b \approx \beta$.*

Proof of Proposition A1: Define the participation set for a contract (s, b) to be

$$P(s, b) = \{\theta \in \Theta | E[w|\theta] \geq \bar{u} + \beta\theta\}.$$

For a given contract (s, b) the expected profits of the firm is

$$E\pi(s, b) = \int_{P(s,b)} E[\pi|\theta] f(\theta) d\theta,$$

where $E[\pi|\theta] = \gamma\theta(1 - b) - s - k$. Now if $b > 1$, $E[\pi|\theta] < 0$. Thus the firm will never set $b > 1$. If $b \leq 1$, $E[\pi|\theta]$ increases in θ, and thus is monotonic in θ.

Consider a contract $(s, b) = (\bar{u}, \frac{\beta}{\gamma})$. Observe that $P(\bar{u}, \beta) = \Theta$, so

$$E\pi(\bar{u}, \beta) = \gamma\mu_\theta(1 - \beta) - \bar{u} - k$$

is the firm's expected profit. Call this contract $(\bar{u}, \beta)$ the fallback contract.

Suppose $b < \beta$. Then either $s < \bar{u}$ or $s \geq \bar{u}$. If $s < \bar{u}$, then $P(s,b) = \emptyset \subset \Theta = P(\bar{u},\beta)$. By monotonicity of expected profit, $E\pi(s,b) < E\pi(s,\beta)$, and thus the fallback contract generates more profit for the firm. Alternatively if $s \geq \bar{u}$, $P(s,b) = [0,\theta^*] \subset \Theta = P(s,\beta)$. By monotonicity of expected profit, $E\pi(s,b) < E\pi(s,\beta)$ and the fallback contract again dominates. In either case, the firm will never select $b < \beta$, since it can always do better with the fallback contract.

Now suppose $b \geq \beta$. If $s > \bar{u}$, then $P(s,b) = \Theta$. But the firm could always do better by lowering the salary to $\bar{u}$, which not affect participation, but increases profits. Formally, $P(\bar{u},b) = \Theta = P(s,b)$ and $\gamma\theta^*(1-b) - \bar{u} - k > \gamma\theta(1-b) - s - k$. Then

$$E\pi(\bar{u},b) = \gamma\mu_\theta(1-b) - \bar{u} - k > \gamma\mu_\theta(1-b) - s - k = E\pi(s,b).$$

Thus, $s \leq \bar{u}$. Therefore, the feasibility set is $FS \equiv \{(s,b)|s \leq \bar{u}, b \geq \beta\}$. Define the efficient set to be all contracts that implement efficient sorting:

$$ES \equiv \{(s,b)|\theta^* = \theta^{FB}, s \leq \bar{u}, b \geq \beta\} \subset FS.$$

Observe that every contract in the efficient set generates expected surplus less expected wage payments. Because every such contract implements θ^{FB}, the contract parameters do not change the variable of integration above, and only lower the expected wage payment. Thus maximizing expected surplus over the efficient set is equivalent to minimizing the expected wage payments:

$$\arg\max_{ES} E\pi(s,b) = \arg\min_{ES} \int_{\theta^{FB}} E[w|\theta]f(\theta)d\theta.$$

Now for every contract in the efficient set, $s = \bar{u} - \gamma\theta^{FB}(b-\beta)$ and $b \geq \beta$. Therefore,

$$E[w|\theta] = s + b\gamma\theta = \bar{u} + b\gamma(\theta - \theta^{FB}) + \gamma\beta\theta^{FB}.$$

is a linear and nondecreasing function in b for all $\theta > \theta^{FB}$. Minimizing this function is equivalent to setting b as low as possible, s.t. $b \geq \beta$.

Let $\eta > 0$. Take $b = \beta + \eta$ and $s = \bar{u} - k\gamma\eta/(m+\gamma)$. Observe that $\theta^*(s,b) = \theta^{FB}$.

Now $E[\pi|\theta] = \gamma\theta - E[w|\theta] - k + m\theta$, where $E[w|\theta] = \bar{u} + b\gamma(\theta - \theta^{FB})$. Setting $b = \eta$ and simplifying,

$$E[\pi|\theta] = (\gamma + m)\theta - \bar{u} - k - \eta\gamma(\theta - \theta^{FB}).$$

This contract attracts only $\theta > \theta^{FB}$, so ex-ante profits are

$$\begin{aligned} E\pi(s,b) &= \int_{\theta^{FB}}^{\infty} E[\pi|\theta]dF. \\ &= \int_{\theta^{FB}}^{\infty} [(\gamma+m)\theta - \bar{u} - k]dF - \eta\gamma \int_{\theta^{FB}}^{\infty} (\theta - \theta^{FB})dF. \end{aligned}$$

The fallback contract generates expected profits

$$\begin{aligned} E\pi(\bar{u},\beta) &= \int_{\Theta} ((\gamma+m)\theta - \bar{u} - k)dF \\ &= \int_{0}^{\theta^{FB}} ((\gamma+m)\theta - k - \bar{u})dF + \int_{\theta^{FB}}^{\infty} ((\gamma+m)\theta - \bar{u} - k)dF \end{aligned}$$

Thus, $E\pi(s,b) > E\pi(s,0)$ if and only if

$$\eta < \frac{-\int_0^{\theta^{FB}} [(\gamma+m)\theta - \bar{u} - k]dF}{\gamma \int_{\theta^{FB}}^{\infty} (\theta - \theta^{FB})dF}$$

The denominator is positive because $\theta > \theta^{FB}$. The numerator is positive by definition of θ^{FB}, since $(\gamma+m)\theta < k$ for $\theta < \theta^{FB}$. Therefore,

$$\lim_{\eta \downarrow 0} E\pi(s,b) > E\pi(\bar{u},0).$$

Thus the optimal contract is $s = \bar{u} - k\gamma\eta/(m+\gamma) \longrightarrow \bar{u}$ and $b = \eta \longrightarrow 0$. Thus $s \approx \bar{u}$ and $b \approx \beta$.

■

7 Renegotiation Proofness

Suppose the firm has the option to renegotiate the contract after the manager accepts it. This means that the firm can re-contract with the manager in the interim stage, ie. the stage after the manager accepts the contract, but before he works and/or output uncertainty is realized. Suppose the firm offers a new contract to the manager that replaces the prior contract. Since employment is voluntary, the manager must now decide whether to accept this new contract. If he does not, he can leave the firm and collect his outside option. Because the firm is the party that writes the original contract, it is also the party that offers the new contract for potential renegotiation. Assume no other change in the information structure; the manager cannot communicate his type to the firm as such communication was not possible before. The fundamental information environment is unchanged; the only new information is that the firm now has some information on the type of the manager, namely, that the firm knows the manager accepted the prior contract.

The firm will renegotiate if it can use its new information to offer a new contract that it finds more profitable. For a given prior contract $(\hat{s}, \hat{b})$, the firm now knows that $\theta > \theta^*(\hat{s}, \hat{b})$. If the firm decides to recontract, it will choose to optimize its expected profits against the conditional density $f(\theta|\theta > \theta^*)$, rather than the unconditional density $f(\theta)$ as it did before. However, maximizing profits against the conditional density yields the same contract as when the firm maximizes against the unconditional density. Intuitively, knowing that the manager accepted the prior contract does not change the firm's objective function on the margin. When offering the new contract, the firm must still balance minimizing its cost of labor, inducing participation, and providing incentives for effort (in the case of Propositions 2 and 3). Formally, the probability that the manager accepted the original contract simply scales the firm's objective function but does not change the marginal cost and benefits from altering the contract. Moreover, a manager who faces this new contract will evaluate his inside opportunities (from employment under this new contract) against his outside options. That he accepted the original contract does not change the manager's new decision. So both the manager's and the firm's problem are the same as before. Thus, the original contract is renegotiation-proof.

To show this, it is necessary to prove that the optimal contracts in Propositions 1, 2, and 3 do not change under renegotiation. The result below examines the firm's and manager's problems in terms of variables such as expected profits, expected wages, and the marginal type. Each of these variables has different manifestations in Propositions 1, 2, and 3, but the variables themselves are general, and hence the proof applies to all three.[1]

Proposition A2 *The optimal contract in Propositions 1, 2, and 3 are renegotiation-proof.*

Proof of Proposition A2: Let $(\hat{s}, \hat{b})$ be the optimal contract. Let $\hat{\theta} = \theta^*(\hat{s}, \hat{b})$, defined in each proposition, according to $E[w|\theta^*] = \bar{u}$. Suppose the manager accepts this contract. The firm therefore knows that $\theta > \theta^*(\hat{s}, \hat{b}) = \hat{\theta}$.

At the interim stage, the firm has the option of writing a new contract (s, b) to replace the old contract. The manager of type $\theta > \hat{\theta}$ can either accept this contract to earn $E[w|\theta]$, or reject this contract and collect his outside option $\bar{u}$. Every such type of manager will accept the new contract if $E[w|\theta] \geq \bar{u}$. The marginal type θ^* is indifferent, so $E[w|\theta^*] = \bar{u}$. Let $\theta^*(s, b)$ denote the marginal type, for the new contract (s, b).

The firm now picks the new contract to maximize profits. The firm will never choose a contract

[1]For example, in Section 2, $\theta^* = \frac{\bar{u}-s}{b\gamma}$, while in Section 3, $\theta^* = \frac{\sqrt{2c(\bar{u}-s)}}{b\gamma}$, but both are defined by the condition $E[w|\theta^*] = \bar{u}$.

that induces a marginal type below $\hat{\theta}$, since such contracts are more expensive (θ^* decreases in (s, b)) and does not change participation ($\hat{\theta}$ is the lowest possible cutoff, since $\theta < \hat{\theta}$ rejected the initial contract, and therefore is not in the pool of available managers). Denote the relevant set of contracts

$$R = \{(s, b) : \theta^*(s, b) \geq \hat{\theta}\}$$

The firm knows that the pool of available managers is now $\theta > \hat{\theta}$, hence it optimizes against the conditional density $f(\theta|\theta > \hat{\theta})$. Thus the firm solves

$$\max_{(s,b) \subseteq R} \int_{\theta^*(s,b)}^{\infty} E[\pi|\theta] f(\theta|\theta > \hat{\theta}) d\theta \tag{30}$$

To solve this, I drop the constraint that $(s, b) \subset R$. If the optimal contract lies within R, then the constraint is not necessary for the maximization. Observe that the conditional density is given by

$$f(\theta|\theta > \theta^*) = \frac{f(\theta)}{E[\theta|\theta > \theta^*]} \equiv \frac{f(\theta)}{1 - F(\theta^*)} \tag{31}$$

Therefore, the firm's optimization simplifies to

$$\max_{(s,b)} \frac{1}{1 - F(\theta^*)} \int_{\theta^*(s,b)}^{\infty} E[\pi|\theta] f(\theta) d\theta \tag{32}$$

Let $(\tilde{s}, \tilde{b})$ be the solution to this optimization. But observe that the fraction outside the integral is simply a scalar, and hence the firm's problem at the interim stage is the same as in the ex-ante stage. Thus, $(\tilde{s}, \tilde{b}) = (\hat{s}, \hat{b})$, and $\theta^*(\tilde{s}, \tilde{b}) = \hat{\theta} = \theta^*(\hat{s}, \hat{b})$. Thus, $(\tilde{s}, \tilde{b}) \subseteq R$, and hence dropping the constraint was justified.

■